The zero set of conformal vector fields

Andrei Moroianu, Liviu Ornea

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ANDREI MOROIANU AND LIVIU ORNEA

Abstract. We show that every connected component of the zero set of a conformal vector field on a Riemannian manifold is totally umbilical.

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1. Introduction

A well-known result of S. Kobayashi, [3], states that the connected components of the set of zeros of a Killing vector field on a Riemannian manifold are totally geodesic submanifolds of even codimension (the case of of isolated singular points is included). This is a purely local result.

In [1], D.E. Blair proved the similar statement for conformal vector fields, the zero set being now a totally umbilical submanifold. However, his proof only works on *compact* manifolds and makes explicit use of the Obata theorem.

In this short note we prove that, in fact, the result is local, like that of Kobayashi. However, we were not able to prove it in full generality. Precisely, we could not prove that each connected component of the zero set of a conformal vector field is always a submanifold. For Killing fields, this follows by identifying this set with the image through the exponential map of a vector subspace of the tangent space. In the conformal case, we could not find a connection whose exponential map play this rôle. We thus need to assume that the zero set contains a differentiable curve. We do not know whether this assumption can be removed or not.

2. Main result

Let ζ be a conformal vector field on a Riemannian manifold (M^n, g) $(n \geq 3)$. We denote by ξ its metric dual 1-form, and by $\varphi := -\delta\xi/n$. Then ξ satisfies:

$$\nabla_Y \xi = \frac{1}{2} Y \lrcorner d\xi + \varphi Y^\flat, \qquad \forall Y \in TM. \tag{1}$$

In [2, Theorem 2.1], Capocci proved the following result:

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Theorem 2.1. [2] *Let x be a zero of the conformal field ζ. Then ζ is a homothetic vector field with respect to some conformal metric $\tilde{g}$ defined on a neighbourhood of x if and only if the gradient of φ with respect to g belongs to the image of $\nabla\zeta$ at x; moreover, ζ is Killing with respect to $\tilde{g}$ if, in addition, $\varphi(x) = 0$.*

Hence, if 2.1 applies, the connected component of the zero set of ζ is totally geodesic with respect to $\tilde{g}$, and hence totally umbilical with respect to g (see, *e.g.* [1, pp.2–3]).

We can now state our result:

Theorem 2.2. *Let ζ be a conformal vector field on a Riemannian manifold (M^n, g) ($n \geq 3$). If a connected component of its zero set contains a smooth curve, then this connected component is a totally umbilical submanifold of M.*

We show that we can apply Theorem 2.1.

From now on we identify 1-forms and vectors via the metric g whenever there is no risk of confusion. We fix some point x on M, and a local orthonormal frame $\{e_i\}$ parallel at x. We also assume that all vector fields X, Y,... are parallel at x in order to simplify the computations. Differentiating (1) in the direction of some vector field X yields

$$\nabla_X \nabla_Y \xi = \frac{1}{2} Y \lrcorner \nabla_X d\xi + X(\varphi)Y, \qquad \forall X, Y \in TM. \tag{2}$$

On the other hand, we also have

$$\nabla_X \nabla_Y \xi = R_{X,Y}\xi + \nabla_Y \nabla_X \xi \overset{(1)}{=} R_{X,Y}\xi + \nabla_Y (\frac{1}{2} X \lrcorner d\xi + \varphi X). \tag{3}$$

We use Einstein's summation convention on repeated subscripts. From (2) and (3) we get

$$\begin{aligned} \nabla_X d\xi &= \frac{1}{2} e_i \wedge (e_i \lrcorner \nabla_X d\xi) \overset{(2)}{=} e_i \wedge \nabla_X \nabla_{e_i} \xi \\ &\overset{(3)}{=} e_i \wedge (R_{X,e_i}\xi + \frac{1}{2} X \lrcorner \nabla_{e_i} d\xi + e_i(\varphi)X) \\ &= e_i \wedge R_{X,e_i}\xi + \frac{1}{2} \nabla_X d\xi + d\varphi \wedge X. \end{aligned}$$

To compute the curvature term we can use the first Bianchi identity:

$$\begin{aligned} e_i \wedge R_{X,e_i}\xi &= e_i \wedge e_j R(X, e_i, \xi, e_j) = e_i \wedge e_j (R(X, e_j, \xi, e_i) + R(X, \xi, e_i, e_j)) \\ &= -e_i \wedge e_j R(X, e_i, \xi, e_j) + 2R_{X,\xi}, \end{aligned}$$

whence $e_i \wedge R_{X,e_i}\xi = R_{X,\xi}$. The previous equation yields

$$\nabla_X d\xi = 2R_{X,\xi} + 2d\varphi \wedge X. \tag{4}$$

Lemma 2.3. *Let Z denote the zero set of ζ. Assume that Z contains a smooth curve $c(t)$ (that we may suppose parametrized by arc length,* i.e. *$g(\dot{c}, \dot{c}) = 1$).*

Then $\nabla_{\dot{c}}\zeta = 0$, $\varphi(c(t)) = 0$ and $\dot{c}(\varphi) = 0$.

Proof. Extend $\dot c$ to a vector field Y on M. Since the flow φ_s of Y through any point of the curve c stays (locally) on the same curve, the Lie derivative of ζ with respect to Y vanishes at every point of c:

$$(\mathcal{L}_Y\zeta)_{c(t)} = -\frac{d}{ds}|_{s=0}((\varphi_s)_*\zeta)_{c(t)} = -\frac{d}{ds}|_{s=0}(\varphi_s)_*(\zeta_{\varphi_s^{-1}c(t))}) = 0.$$

We thus get at $c(t)$

$$\nabla_{\dot c}\zeta = \nabla_Y\zeta = \nabla_\zeta Y = 0.$$

Moreover, since $g(\nabla_{\dot c}\zeta, \dot c) = \varphi g(\dot c, \dot c) = \varphi$, we must have $\varphi(c(t)) = 0$. Differentiating this with respect to t yields the last relation.

□

Taking (1) into account we get $\dot c \lrcorner d\xi = 0$. We differentiate this relation with respect to $\dot c$ and obtain at any point of c:

$$\begin{aligned} 0 &= \nabla_{\dot c}(\dot c \lrcorner d\xi) = \nabla_{\dot c}\dot c \lrcorner d\xi + \dot c \lrcorner \nabla_{\dot c} d\xi \\ &\overset{(4)}{=} 4\nabla_{\dot c}\dot c \lrcorner \nabla\xi + 2R_{\dot c,\zeta}\dot c + \dot c(\varphi)\dot c - 2d\varphi g(\dot c, \dot c) \\ &= 4\nabla_{\dot c}\dot c \lrcorner \nabla\xi - 2d\varphi. \end{aligned}$$

This shows in particular that $d\varphi$ (more precisely, the gradient of φ with respect to g) belongs to the image of $\nabla\zeta$ at $c(0)$.

According to Theorem 2.1, this completes the proof.

References

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Centre de Mathémathiques, Ecole Polytechnique, 91128 Palaiseau Cedex, France

E-mail address: am@math.polytechnique.fr

Univ. of Bucharest, Faculty of Mathematics, 14 Academiei str., 70109 Bucharest, Romania, and Institute of Mathematics "Simion Stoilow" of the Romanian Academy, 21, Calea Grivitei str., 010702-Bucharest, Romania.

E-mail address: lornea@gta.math.unibuc.ro, Liviu.Ornea@imar.ro